Creating Cosmological Initial Conditions for Fun and Profit

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Abstract

This is a short 'how-to' summary of one method of creating cosmological initial conditions using the Zel'Dovich approximation. Step-by-step instructions are presented for the creation of initial conditions for a simulation based Cartesian grid, with baryonic and dark matter components. A Gaussian distribution of initial density fluctuations is assumed. Relevant equations are contained in an appendix for convenience.

1 Introduction

Creating a set of initial conditions for simulations of cosmological structure formation is, on the surface, a very straightforward task. One specifies a background cosmological model, typically described as a spatially flat or open Robertson-Walker spacetime. Following that, perturbations are imposed upon this background. In reality, numerous approximations must be made which should be carefully considered, and are discussed in some detail below.

The specification of background cosmology requires several inputs: The amount and nature of dark matter, the Hubble parameter H_0, and possibly the amount of baryonic matter and cosmological constant in the universe.

At the epoch of baryon-photon decoupling ($z \sim 1100$), small-amplitude ("linear") fluctuations in density are already present in all of the components that contribute to the energy density of the universe (such as baryons, dark matter, photons, and neutrinos). The statistical nature of these fluctuations depends on their origin. There are two general classes of early universe models that are considered to provide reasonable mechanisms for perturbations: Topological defects (Durrer et. al 2002) and inflation (Guth 1981). Inflation predicts Gaussian fluctuations and defect models are non-Gaussian.

Gaussian fluctuations are simple since they are specified completely by a single function, the power spectrum $P(k)$. In Gaussian models the perturbations are set down immediately (possibly in the pre-inflationary epoch) and evolve in a straightforward manner. In real space, the probability distribution of density fluctuations is a multidimensional Gaussian, and it is very easy to sample a Gaussian random field by sampling its Fourier components on a Cartesian lattice, which is the technique that will be discussed in this paper. For more information on other methods, see the Bertschinger's (1998) review on simulations of cosmological structure formation.

Non-Gaussian models are much more complicated to model. Not only do they require more initial information than a simple power spectrum, they also are more costly in a computational sense. Typically, topological defects induce matter density fluctuations from the time of their creation in the early universe to the present day, and the dynamics of their formation and evolution are relativistic and nonlinear. For more information on creating initial conditions of topological defects, see Bertschinger (1998) or Durrer et. al (2002).

2 Creating Gaussian Random Fields

The creation of cosmological initial conditions using a Gaussian random field is relatively straightforward. Given a power spectrum P(k), the linear density fluctuation field is calculated at some initial time (typically z $\sim$ 100 for high-resolution simulations). From this, dark matter particle positions and velocities are determined, along with baryon density and velocity fields. These steps are described further in Sections 2.1 and 2.2.

2.1 The Linear Density Fluctuation Field

The first step towards creating a Gaussian density field is to specify a power spectrum, which is a function that relates amplitudes of density perturbations with wavelength λ to their wavenumber k, where $k \equiv 2\pi/\lambda$. The power spectrum of the fractional density fluctuations at the redshift z = z_{eq} when the energy density in matter is equal to that in radiation, can be related to the primordial power spectrum by $P(k, z_{eq}) \sim T^2(k)P_p(k)$, where T(k) is the matter transfer function as a function of wave number and describes the processing of the initial density perturbations during the radiation dominated era (Padmanabhan 1993) and P_p(k) is the primordial matter power spectrum, which typically has a power law form, ie, P_p(k) $\sim$ k^n, where n is the index of the primordial power specturm. This index is equal to unity for Harrison-Zel'Dovich scale-invariant spectra, a typical model. (Harrison 1970; Zel'Dovich 1972) The power spectrum at any redshift z in the matter dominated era may then be written in the form

$$\frac{k^3}{2\pi^2}P(k,z) = \left(\frac{ck}{H_0}\right)^{3+n} \delta_H^2 T^2(k) D_g^2(z)/D_g^2(0), \tag{1}$$

where the D_g's are the linear grown factor for perturbations, which is defined in Peebles (1980) and is also in Appendix A. A closed-form fitting function (much more appropriate for computation) is given in Eisenstein & Hu (1999). δ_H is a constant describing the amplitude of density fluctuation, which can be provided from observations of the CMB or from large scale structure, or can be normalized by comparing to, eg, σ_8, which is the rms amplitude of the mass fluctuations in an the universe when smoothed using a top-hat function with characteristic radius of 8 h^-1 Mpc.

Once P(k) has been determined, we then proceed to calculate δ_k, namely, the density fluctuations in k-space. To simplify matters, we choose a cubic three-dimensional Cartesian grid with N grid points per dimension. Each of the grid points has a unique identifier (n_x,n_y,n_z) associated with its location along the (x,y,z) axis. We sample the power spectrum P(k) discretely at each grid location (n_x,n_y,n_z), obtaining k in this manner:

$$k^2 = (n_x^2 + n_y^2 + n_z^2)dk^2 \tag{2}$$

with dk = $2\pi/L_{box}$, and L_{box} is the size of the simulation box in megaparsecs. It is important to note that the method of sampling P(k) at discrete points, while convenient when doing simulations on a Cartesian grid, assumes that P(k) is a smoothly-varying (and slowly varying) function of k. If this is NOT true, errors will be introduced unless P(k) is sampled in a different way. The k-space fluctuation δ_k is a complex value with random phase and amplitude, where the distribution of amplitudes fit a Gaussian distribution with a mean of P(k). One method to calculate it is to generate a phase angle θ, which is randomly selected in a uniform manner in the interval $[0, 2\pi]$, and an amplitude A such that

$$A = \sqrt{-log(R) * P(k)} \tag{3}$$

where R is randomly selected in a uniform manner in the interval (0,1). δ_k is then

$$\delta_k = Ae^{i\theta} \tag{4}$$

We then perform a Fourier transform on the grid of δ_k values, giving δ_x, the relative density fluctuation at each spatial grid point in the simulation volume. The actual physical (meaning, in position space rather than k-space) dark matter density is then

$$\rho_{DM}(\vec{x}) = (1 + \delta_x)\overline{\rho}_{DM} \tag{5}$$

where $\overline{\rho}_{DM}$ is the mean dark matter density in the simulation. The perturbations in the dark matter and baryon densities are assumed to be coupled, which is true in the linear regime, so the baryon density at any position is

$$\rho_b(\vec{x}) = \frac{\Omega_b}{\Omega_{DM}}\rho_{DM}(\vec{x}) \tag{6}$$

where Ω_b and Ω_{DM} are the ratios of the present-day mean baryon and dark matter densities to the critical density of the universe, ie, $\Omega_b = \rho_b/\rho_c$.

2.2 Position and Velocity Fields

In cosmological simulations, dark matter is usually represented by discrete particles rather than a uniform grid of densities. In that case, these particles must be perturbed so as to reflect the density fluctuations described in Section 2.1. The standard approach for the dark matter is to displace equal-mass particles from a uniform Cartesian lattice using the Zel'Dovich (1970) approximation:

$$\vec{x} = \vec{q} + D(t)\psi(\vec{q}) \tag{7}$$

and

$$\vec{v} = a\frac{dD}{dt}\vec{\psi} = aHfD\vec{\psi}, \tag{8}$$

where $\vec{q}$ labels the unperturbed lattice position, D(t) is the growth factor of the linear growing mode, and $f = dlnD/dlna$ is its logarithmic growth rate (Peebles 1980). The irrotational (curl-free) displacement field $\vec{\psi}$ is computed by solving the linearized continuity equation,

$$\vec{\nabla} \cdot \vec{\psi} = -\frac{\delta_x}{D(t)}, \tag{9}$$

Since the equation is linearized, $\vec{\psi}$ can be found by taking the Fourier transform of $-i\delta_k \hat{k}/kD(k)$.

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A Some useful equations

A useful fitting function for the linear growth function D_g(z) given by Eisenstein & Hu (1999) as

$$D_g(z) = \frac{1}{1+z}\frac{5\Omega_m(z)}{2}\left\{\Omega_m(z)^{4/7} - \Omega_\Lambda(z) + \left[1 + \frac{\Omega(z)}{2}\right]\left[1 + \frac{\Omega_\Lambda(z)}{70}\right]\right\}^{-1} \tag{10}$$

where

$$\Omega_m(z) = \Omega_{m,0}(1+z)^3 g^{-2}(z), \tag{11}$$

$$\Omega_\Lambda(z) = \Omega_{\Lambda,0} g^{-2}(z), \tag{12}$$

and

$$g(z) = \Omega_{m,0}(1+z)^3 + (1 - \Omega_{m,0} - \Omega_{\Lambda,0})(1+z)^2 + \Omega_{\Lambda,0} \tag{13}$$

$\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ are the matter and vacuum energy density with respect to the critical density, $\rho_c = 3H_0^2/8\pi G$, at the present epoch.

The scale factor a(t) is calculated for any epoch by solving the Friedman equation,

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \tag{14}$$

which comes from the G_0^0 tensor component of Einstein's equation. ρ is actually a sum of all contributions to the energy density at the current epoch, which include, but are not limited to, baryons, dark matter, radiation, and neutrinos (massive or massless). These quantities do not all vary in density the same way with the scale factor - for example, the dark matter density varies with the scale factor as $\rho_{DM} \sim a^{-3}$, while radiation scales as $\rho_r \sim a^{-4}$. This equation can be rewritten as:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = H_0^2\left(\frac{\Omega_{r,0}}{a^3} + \frac{\Omega_{m,0}}{a^3} + \Omega_\Lambda\right) \tag{15}$$

In general an exact solution of this can only be obtained numerically.

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